excitation. These results appear generally to agree with the data previously presented.

Appendix

Ordinary Coherence Function for Perfectly Correlated Excitation Field

From Ref. 5,

$$\{S_{xy}(f)\} = [S_{xx}]\{H_{xy}\}$$

Let point $x = (0,0)$ be the excitation point for the first element of the column matrices in this expression. Then, in view of Eq. (9b), the first element of the left-hand matrix is

$$S_{0y}(f) = \sum_{i=1}^{N} S_{0i}H_{iy} = \sum_{i=1}^{N} k_{x_i}H_{iy}S_{00}(f)$$

Combining this result along with Eqs. (11) and (12), it is found that

$$\gamma_{0y}^2 = |\bar{S}_{0y}(f)|^2/[S_{yy}(f)\bar{S}_{00}(f)] = 1$$

The validity of this expression is a joint indication of the linearity of the system as well as an indication of the validity of Eq. (9b) for representing the experimental excitation field.

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MARCH 1971 AIAA JOURNAL VOL. 9, NO. 3

A Theory of the High-Aspect-Ratio Jet Flap

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The method of matched asymptotic expansions, which Van Dyke used to formulate a lifting-line theory for high-aspect-ratio wings, is applied to a wing with jet flap. The development differs from Van Dyke's in that velocity components instead of the velocity potential are the dependent variables, thin-airfoil approximations are used throughout, and the jet flap is present. The theory is given for the case of a flat elliptic wing with spanwise-uniform jet-momentum coefficient and jet angle and the result is a simple equation for the lift coefficient. Comparison with the results of two earlier finite-aspect-ratio jet-flap theories shows close agreement. Certain approximations needed in the earlier theories to solve the integral equation for the upwash induced by a semi-infinite lifting surface are avoided by consistent use of the principle that, in the limit as the aspect ratio becomes infinite, the change in lift due to induced incidence is much smaller than the lift, so that the integral equation does not have to be solved.

I. Introduction

A JET flap is formed by a high-speed jet which emerges from a wing through a thin slit along its trailing edge. It affects the lift both by direct momentum reaction on the internal ducting and by changing the pressure distribution on the outer surface of the wing. Since this can increase the lift while most of the jet thrust is recovered as propulsion, the jet flap represents a way of combining the lifting and propulsive systems of jet aircraft.

An analytical model for a two-dimensional wing with jet flap has been developed by Spence[1,2] and others using the principles of thin-airfoil theory. The flows inside and outside the jet are assumed to be irrotational and at constant densities. The wing and jet are represented by vorticity bound to a semi-infinite line downstream of the leading edge on which mixed boundary conditions apply. The lift and pitching-moment coefficients for steady flow past an uncambered wing were calculated numerically in Ref. 1 and the lift coefficient was calculated analytically in Ref. 2. Erickson calculated the pitching-moment coefficient analytically in Ref. 3.

The three-dimensional analog of this jet-flap representation is a horseshoe-vortex system which lies in a semi-infinite strip of width equal to the wing span and with the bound segments of some of the vortices located downstream of the trailing edge. Theories developed by Maskell and Spence[4] and Hartunian[5] use the integral equation for the upwash induced by this vortex system and require certain approximations in order to make the equation tractable. The upwash far downstream—which must be known in order to calculate the lift coefficient—was found to be indeterminate

Received February 16, 1970; revision received June 29, 1970. This paper is based on part of a Ph.D. thesis. The advice and encouragement which W. R. Sears provided while the research reported here was being conducted are gratefully acknowledged. The research was partially supported by the U.S. Air Force Office of Scientific Research under contract number AF 49(638)-1346. The author is now at the Naval Ship Research and Development Center, Washington, D.C.

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in Ref. 4 and double-valued in Ref. 5. Further approximations, an arbitrarily assumed jet shape and the postulate of additional bound vorticity on the jet, were needed in the respective theories.

It is shown in this paper and in the thesis[6] on which it is based that the method of matched asymptotic expansions, with the inverse of the aspect ratio as the small parameter, facilitates the analysis of the high-aspect-ratio wing with jet flap. Van Dyke[7] has used this method to formulate a lifting-line theory which, after a Shanks transformation,[8] reproduces and extends the classical Prandtl result yet avoids the integral equation. In the outer limit Van Dyke found that the wing shrank to a lifting line and in the inner limit the flow near the wing is described by wing-section theory. An advantage of using this method for the wing with jet flap is that the integral equation for the induced upwash does not appear.

II. Analysis

The theory of the jet flap in two dimensions is derived by assuming that the flows inside and outside the jet are irrotational and at constant densities, the wing and jet are thin, and the jet is at high speed. On this basis Spence[1,2] was able to formulate and solve the boundary-value problem for the two-dimensional jet flap as one with mixed boundary conditions on a semi-infinite line parallel to the undisturbed flow. α, the wing angle of incidence, and τ, the angle with which the jet emerges from the wing at the trailing edge, are specified and a boundary condition on the jet is obtained by combining the derivative of the kinematic condition of flow tangency to the jet with the dynamic condition of balance between the pressure difference across the jet and the rate of momentum flow per unit span through the jet divided by its radius of curvature. Spence found the two-dimensional lift coefficient $C_L^{(2)}$ in the form

$$C_L^{(2)} = \alpha(\partial C_L^{(2)}/\partial\alpha) + \tau(\partial C_L^{(2)}/\partial\tau) \tag{1}$$

where the two-dimensional lift derivatives $\partial C_L^{(2)}/\partial\alpha$ and $\partial C_L^{(2)}/\partial\tau$ are functions of the jet coefficient C_J, which is defined as the ratio of the jet momentum-flow rate per unit span divided by the product of the wing chord and the dynamic pressure of the undisturbed flow. Asymptotic series for the lift derivatives in powers and logarithms of C_J are given in Ref. 2. Although these series are only valid for C_J less than 4, they cover the range of engineering interest since C_J is seldom greater than 2 in practice.

The three-dimensional theory retains the assumptions of the two-dimensional theory described previously and requires that the aspect ratio A, defined as the ratio of the span to the mean chord, be high and that there be no appreciable sweepback. The wing span is of unit length and the undisturbed flow has unit speed. An orthogonal coordinate system x, y, z is defined with origin at the center of the wing, x in the direction of the undisturbed flow, y positive to starboard, and z positive up. Respective velocity components parallel to these axes are u, v, and w.

The theory must be valid in the limit as A becomes infinite so a dimensionless chord variable $c(y)$ is used which remains finite in this limit. Thus, $c(y)$ will be defined as the ratio of the local chord to the mean chord. The wing is a flat elliptical plate with major axis in the y direction so $c(y)$ is given by

$$c(y) = (4/\pi)(1 - 4y^2)^{1/2} \tag{2}$$

for $-\frac{1}{2} < y < \frac{1}{2}$. The τ is independent of y and the jet momentum-flow rate/unit span is proportional to $(1 - 4y^2)^{1/2}$ so C_J is independent of y. The jet as well as the wing can support a pressure difference so $\gamma(x,y)$, the circulation/unit length of the bound vorticity and defined by $\gamma(x,y) = u(x,y,0+) - u(x,y,0-)$, can be nonzero for all x downstream of the leading edge and $-\frac{1}{2} < y < \frac{1}{2}$.

The wing, trailing-edge, and jet boundary conditions are $w = -\alpha$ for $-c(y)/2A < x < c(y)/2A$, $w = -\alpha - \tau$ for $x = c(y)/2A$, and $\gamma = [c(y)C_J/2A]\partial w/\partial x$ for $c(y)/2A < x < \infty$. These apply for $-\frac{1}{2} < y < \frac{1}{2}$ and $z = 0$. The constant density and irrotationality result in zero divergence and curl of the velocity vector as the flow equations. Far from the wing the velocity vector approaches its undisturbed value so $(u,v,w) \to (1,0,0)$ as $x^2 + y^2 + z^2 \to \infty$. An important exception to this is on the jet sheet, where its finite span allows it to have a nonzero ultimate deflection angle while the lift on the wing is finite.

If the limit process $A \to \infty$ is applied to this three-dimensional jet-flap boundary-value problem the wing shrinks to a line segment and γ is zero downstream of it. Since the outer expansion can be invalid on this line segment the one-term outer expansion is found immediately as

$$(u_1,v_1,w_1) = (1,0,0) \tag{3}$$

The inner expansion is valid where the outer expansion is not and is an asymptotic solution, for $A \to \infty$, to the full boundary-value problem rewritten in inner variables, which are denoted by capital letters when they differ from the outer variables. The inner independent variables are the outer independent variables x, y, and z stretched by functions of A chosen to make them of order one, as $A \to \infty$, in the region where the outer expansion is invalid. The wing boundary condition given previously shows that the appropriate function for x is A itself and the theory of singular perturbations[7] requires that when the singular region is a line or line segment, the two coordinates normal to it should be stretched by the same factor whereas the one parallel to it should be left unstretched. Therefore the inner independent variables are X, y, and Z with X and Z given by $(X,Z) = A(x,z)$.

The inner dependent variables U, V, W, and Γ are functions of the inner independent variables but none are themselves stretched; thus, they are given by $(U,V,W,\Gamma) = (u,v,w,\gamma)$.

When the flow equations are rewritten in inner variables they are

$$(\partial U/\partial X) + (1/A)(\partial V/\partial y) + (\partial W/\partial Z) = 0 \tag{4}$$

$$(1/A)(\partial W/\partial y) - (\partial V/\partial Z) = 0 \tag{5}$$

$$(\partial U/\partial Z) - (\partial W/\partial X) = 0 \tag{6}$$

and

$$(\partial V/\partial X) - (1/A)(\partial U/\partial y) = 0 \tag{7}$$

and are satisfied for all X, y, and Z except $-c(y)/2 < X < \infty$, $-\frac{1}{2} < y < \frac{1}{2}$, $Z = 0$. The wing, trailing edge, and jet boundary conditions become $W = -\alpha$ for $-c(y)/2 < X < c(y)/2$, $W = -\alpha - \tau$ for $X = c(y)/2$, and $\Gamma = [c(y)C_J/2]\partial W/\partial X$ for $c(y)/2 < X < \infty$. The boundary conditions far from the wing are satisfied by the outer expansion and are not applied to the inner expansion. Instead the method of matched asymptotic expansions requires that the outer and inner expansions be related through their respective inner and outer expansions. [The inner (outer) expansion of the outer (inner) expansion is formed by rewriting it in inner (outer) variables and holding them fixed while expanding for $A \to \infty$.] The form of this asymptotic matching principle that will be used here is[7]

the m-term inner expansion of the (n-term outer expansion) = the n-term outer expansion of the (m-term inner expansion) (8)

The one-term inner expansion is denoted by U_1, V_1, W_1, and Γ_1. Use of Eq. (8) with $n = m = 1$ to match it to the one-term outer expansion results in

$$\lim_{A\to\infty} [U_1(Ax,y,Az),V_1(Ax,y,Az),W_1(Ax,y,Az)] = (1,0,0)$$

or

$$\lim_{X^2+Z^2\to\infty} U_1(X,y,Z) = 1 \quad (9)$$

$$\lim_{X^2+Z^2\to\infty} V_1(X,y,Z) = 0 \quad (10)$$

and

$$\lim_{X^2+Z^2\to\infty} W_1(X,y,Z) = 0 \quad (11)$$

when rewritten in inner variables.

Substitution of U_1, V_1, and W_1 into Eqs. (5) and (7) and application of the limit process $A \to \infty$ show that V_1 is independent of X and Z. Then Eq. (10) shows that V_1 must be zero and Eqs. (4) and (6) show that U_1 and W_1 satisfy the two-dimensional flow equations in X and Z.

In two-dimensional jet-flap theory the ultimate jet-deflection angle is zero, since otherwise an infinite amount of fluid/units span and time would be transferred from beneath the jet to above it to result in infinite lift per unit span on the wing. Thus, Eq. (11) permits two-dimensional jet-flap theory to be used to find the one-term inner expansion. Equation (9) and the wing, trailing-edge, and jet boundary conditions show that the coefficient of lift per unit span is $C_L^{(2)}$ as specified in Eq. (1).

It is shown in Ref. 2 that $C_L^{(2)}$ is related to W_1 in the same way as in conventional two-dimensional wing theory:

$$C_L^{(2)} = -[4\pi/c(y)] \lim_{X\to\infty}[XW_1(X,y,0)]$$

and this can be expressed in outer variables as

$$\lim_{A\to\infty}[AxW_1(Ax,y,0)] = -c(y)C_L^{(2)}/4\pi$$

Therefore the one-term outer expansion (which is of order one in $1/A$) of $W_1(X,y,0)$ is zero, as is required by Eq. (11), and its two-term outer expansion is $-c(y)C_L^{(2)}/4\pi Ax$ or

$$-[c(y)/4\pi]C_L^{(2)}/X \quad (12)$$

when rewritten in inner variables.

The asymptotic matching principle, Eq. (8), with $n = 2$ and $m = 1$ requires that the two-term outer expansion have a one-term inner expansion that matches Expression (12). The order, in $1/A$, of Expression (12) written in outer variables suggests that the next term in the outer expansion is of order $1/A$ so its velocity vector will be denoted by $(1 + u_2/A, v_2/A, w_2/A)$. Since, as $A \to \infty$, the outer expansion is invalid on $x = 0, -\frac{1}{2} < y < \frac{1}{2}, z = 0$ it can be represented by solutions to the flow equations which are singular on this line segment. The weakest singularities which have the antisymmetry in z of the one-term inner expansion are vortices bound to the line segment. Free trailing vortices are associated with them and these will be assumed to lie on $0 < x < \infty, -\frac{1}{2} < y < \frac{1}{2}, z = 0$. If K_2/A is the circulation of the bound vorticity the upwash is given by an equation of classical lifting-line theory:

$$\frac{w_2(x,y,0)}{A} = \frac{1}{4\pi A}\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{dK_2(y')}{dy'} \times \left\{1 + \frac{[x^2 + (y'-y)^2]^{1/2}}{x}\right\}\frac{dy'}{y'-y} \quad (13)$$

The inner expansion for the upwash is found by rewriting Eq. (13) in inner variables and expanding for large A. Thus,

$$\frac{w_2(X/A,y,0)}{A} = \frac{1}{4\pi A}\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{dK_2(y')}{dy'} \times \left\{\frac{1}{y'-y} + \frac{A}{X}\frac{|y'-y|}{y'-y}\left[1 + \frac{X^2}{A^2(y'-y)^2}\right]^{1/2}\right\} dy' =$$
$$\frac{1}{4\pi X}\left[-\int_{-\frac{1}{2}}^{y}\frac{dK_2(y')}{dy'}dy' + \int_{y}^{\frac{1}{2}}\frac{dK_2(y')}{dy'}dy'\right] + \frac{1}{4\pi A}\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{dK_2(y')}{dy'}\frac{dy'}{y'-y} + O\left(\frac{1}{A^2}\right) =$$
$$-\frac{K_2(y)}{2\pi X} - \frac{\alpha_i(y)}{A} + O\left(\frac{1}{A^2}\right)$$

where

$$\alpha_i(y) = -\frac{1}{4\pi}\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{dK_2(y')}{dy'}\frac{dy'}{y'-y} \quad (14)$$

Therefore the one-term and two-term inner expansions of $w_2(x,y,0)/A$ are

$$-K_2(y)/2\pi X \quad (15)$$

and

$$-[K_2(y)/2\pi X] - [\alpha_i(y)/A] \quad (16)$$

If the equation for the x component of velocity induced by $K_2(y)/A$ were written out and its inner expansion calculated it would be found that the two-term inner expansion of $u_1 + u_2/A$ is 1.

Application of Eq. (8) with $n = 2$ and $m = 1$ to Expressions (12) and (15) shows that

$$K_2(y) = c(y)C_L^{(2)}/2 \quad (17)$$

and then Eqs. (14), (17), and (2) give

$$\alpha_i(y) = C_L^{(2)}/\pi \quad (18)$$

The next application of Eq. (8) will require that the two-term outer expansion of the two-term inner expansion of $W(X,y,0)$ match Expression (16) so the two-term inner expansion will be written as $U_1 + U_2/A$, V_2/A, and so on. Substitution of the velocity components into Eqs. (4) and (6) shows that $U_1 + U_2/A$ and $W_1 + W_2/A$ satisfy the same two-dimensional flow equations that U_1 and W_1 did. Substitution of $W_1 + W_2/A$ and $\Gamma_1 + \Gamma_2/A$ into the wing, trailing edge, and jet boundary conditions shows that they satisfy the same ones that W_1 and Γ_1 did. Matching conditions on $U_1 + U_2/A$ and $W_1 + W_2/A$ are provided by Eq. (8) with $n = m = 2$. Since the two-term inner expansion of $u_1 + u_2/A$ is 1 the matching condition on $U_1 + U_2/A$ is

$$\lim_{A\to\infty}[U_1(Ax,y,Az) + U_2(Ax,y,Az)/A] = 1$$

or, in inner variables,

$$\lim_{X^2+Z^2\to\infty}[U_1(X,y,Z) + U_2(X,y,Z)/A] = 1$$

Expression (16) and Eq. (17) show that the matching condition on $W_1 + W_2/A$ is

$$\lim_{A\to\infty}[W_1(Ax,y,0) + W_2(Ax,y,0)/A] = -[c(y)C_L^{(2)}/4\pi Ax] - \alpha_i/A$$

or

$$\lim_{X\to\infty}[W_1X(,y,0) + W_2(X,y,0)/A] = -\frac{c(y)C_L^{(2)}}{4\pi X} - \frac{\alpha_i}{A} \quad (19)$$

Equation (19) shows that $W_1 + (W_2 + \alpha_i)/A$ rather than $W_1 + W_2/A$ has the zero ultimate jet-deflection angle needed to apply two-dimensional jet-flap theory. Use of $W_1 + (W_2 + \alpha_i)/A$ in place of $W_1 + W_2/A$ does not affect the two-dimensional flow equations or the jet boundary condition but does require that α be replaced by $\alpha - \alpha_i/A$ in the wing and trailing-edge boundary conditions in order for $W_1 + W_2/A$ to satisfy the correct ones. Thus, $-\alpha_i/A$ is the induced incidence of classical lifting-line theory.

The two-dimensional jet-flap theory gives $C_L^{(2)} - (\alpha_i/A)\partial C_L^{(2)}/\partial\alpha$, with $C_L^{(2)}$ as specified by Eq. (1), as the lift coefficient which satisfies

$$-\frac{4\pi}{c(y)}\lim_{X\to\infty}\{X[W_1(X,y,0) + W_2(X,y,0)/A + \alpha_i/A]\} = C_L^{(2)} - (\alpha_i/A)\partial C_L^{(2)}/\partial\alpha$$

Therefore

$$\lim_{A\to\infty}\{Ax[W_1(Ax,y,0) + W_2(Ax,y,0)/A]\} = -\frac{c(y)}{4\pi}\left(C_L^{(2)} - \frac{\alpha_i}{A}\frac{\partial C_L^{(2)}}{\partial\alpha}\right) - x\alpha_i$$

and the three-term outer expansion of $W_1(X,y,0) + W_2(X,y,0)/A$ is

$$-\frac{1}{A}\left[\alpha_i + \frac{c(y)}{4\pi x}C_L^{(2)}\right] + \frac{1}{A^2}\frac{c(y)}{4\pi x}\alpha_i\frac{\partial C_L^{(2)}}{\partial\alpha}$$

or

$$-\frac{\alpha_i}{A} - \frac{c(y)}{4\pi X}\left(C_L^{(2)} - \frac{\alpha_i}{A}\frac{\partial C_L^{(2)}}{\partial\alpha}\right) \quad (20)$$

when rewritten in inner variables.

The order of the lift-derivative term in Expression (20) written in outer variables suggests that the next term in the outer expansion is of order $1/A^2$. Thus, the circulation of the bound vorticity will be written as $K_2(y)/A + K_3(y)/A^2$. The relation between it and the upwash is

$$\frac{w_2(x,y,0)}{A} + \frac{w_3(x,y,0)}{A^2} = \frac{1}{4\pi A}\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{d}{dy'}\times \left[K_2(y') + \frac{K_3(y')}{A}\right]\left\{1 + \frac{1}{x}[x^2 + (y'-y)^2]^{1/2}\right\}\frac{dy'}{y'-y} \quad (21)$$

Application to Eq. (21) of the procedure which gave Expressions (15) and (16) and use of Eqs. (14) and (17) show that the two-term inner expansion of $w_2(x,y,0)/A + w_3(x,y,0)/A^2$ is

$$-c(y)C_L^{(2)}/4\pi X - [\alpha_i + K_3(y)/2\pi X]/A.$$

Use of Eq. (8) with $n = 3$ and $m = 2$ to match this to Expression (20) results in

$$K_3(y) = -[c(y)/2]\alpha_i\partial C_L^{(2)}/\partial\alpha \quad (22)$$

It is shown in Refs. 4 and 5 that the lift coefficient C_L, defined as the ratio of the lift to the product of the wing planform area and the dynamic pressure of the undisturbed flow, is given by

$$C_L = -[C_J + (\pi A/2)]w_\infty \quad (23)$$

for this case where α, τ, and C_J are independent of y and $c(y)$ is given by Eq. (2). In general w_∞ depends on y and is defined by

$$w_\infty(y) = \lim_{x\to\infty} w(x,y,0)$$

and so is proportional to the ultimate jet-deflection angle. Equation (21) gives

$$w_\infty(y) = \frac{1}{2\pi A}\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{d}{dy'}\left[K_2(y') + \frac{K_3(y')}{A}\right]\frac{dy'}{y'-y} + o(1/A^2)$$

and Eqs. (17), (22), and (18) give

$$w_\infty(y) = \frac{C_L^{(2)}}{4\pi A}\left(1 - \frac{1}{\pi A}\frac{\partial C_L^{(2)}}{\partial\alpha}\right)\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{dc(y')}{dy'}\frac{dy'}{y'-y} + o(1/A^2)$$

so, with Eq. (2),

$$w_\infty = -\frac{2C_L^{(2)}}{\pi A}\left(1 - \frac{1}{\pi A}\frac{\partial C_L^{(2)}}{\partial\alpha}\right) + o\left(\frac{1}{A^2}\right) \quad (24)$$

Equation (24) is, except for the dependence of $C_L^{(2)}$ and $\partial C_L^{(2)}/\partial\alpha$ on C_J, the expression for the ultimate upwash of a flat wing without jet flap and with planform given by Eq. (2).

Equations (23) and (24) show that the lift coefficient is given by

$$C_L = C_L^{(2)}\left(1 + \frac{2C_J}{\pi A}\right)\left(1 - \frac{1}{\pi A}\frac{\partial C_L^{(2)}}{\partial\alpha}\right) + o\left(\frac{1}{A}\right) \quad (25)$$

The final expression for C_L is obtained by applying the form of the Shanks Transformation[8] used successfully in Ref. 7. This transforms an expansion of the form $1 + a\epsilon + o(\epsilon)$, for $\epsilon \to 0$, to $1/(1 - a\epsilon)$. The result of applying it to Eq. (25) is

$$C_L/C_L^{(2)} = \pi A/[\pi A + (\partial C_L^{(2)}/\partial\alpha) - 2C_J] \quad (26)$$

for the ratio of the three- and two-dimensional lift coefficients.

III. Comparison with other Theories

The equation for the ratio of lift coefficients given by Maskell and Spence in Ref. 4 is

$$C_L/C_L^{(2)} = [\pi A + 2C_J]/[\pi A + 2(\partial C_L^{(2)}/\partial\alpha) - 2\pi(1+\sigma)] \quad (27)$$

where σ is defined as 1 + 2 (induced downwash at wing)/(ultimate upwash). Therefore $\sigma = 1 + 2\alpha_i/Aw_\infty + o(1/A)$ and Eqs. (18) and (24) enable σ to be shown to be o(1) in $1/A$ so its contribution to Eq. (27) is $o(1/A)$ and, since Eq. (26) neglects $o(1/A)$, σ will be neglected here. In Ref. 4 it is found that σ is a small positive quantity which depends on α and τ as well as C_J and A so an advantage of neglecting it in Eq. (27) is that the ratio of lift coefficients is then independent of wing incidence and jet angle.

With σ neglected Eq. (27) differs from Eq. (26) in the presence of $+ 2C_J$ in the numerator instead of $-2C_J$ in the denominator, and in the presence of $2\partial C_L^{(2)}/\partial\alpha - 2\pi$ instead of $\partial C_L^{(2)}/\partial\alpha$ in the denominator. Although Eq. (25) contains the term $1 + 2C_J/\pi A$ application of the Shanks transformation requires that $2C_J/\pi A$ be reversed in sign and moved to the denominator of Eq. (26). In Ref. 2 Spence found that, as $C_J \to 0$, $(1/2\pi)\partial C_L^{(2)}/\partial\alpha = 1 + (C_J/4\pi)\log(16e^\gamma/C_J) + O(C_J)$ where γ is Euler's constant. (This expression is valid for $C_J < 4$ so the second term on the right-hand side is positive.) Therefore, $2\partial C_L^{(2)}/\partial\alpha - 2\pi = \partial C_L^{(2)}/\partial\alpha + (C_J/2)\log(16e^\gamma/C_J) + O(C_J)$ so, in the limit as $A \to \infty$ and $C_J \to 0$, Ref. 4 gives a ratio of lift coefficients smaller by $(C_J/2\pi A)\log(16e^\gamma/C_J)$ in the denominator than the ratio of the present theory.

In Ref. 5 Hartunian found that the ratio of lift coefficients is given by

$$\frac{C_L}{C_L^{(2)}} = \frac{\pi A + 2C_J}{\pi A + (\partial C_L^{(2)}/\partial\alpha) - 8\pi D_0} \quad (28)$$

D_0 is given as a function of C_J in Table 1 and Fig. 11 of Ref. 5, where it is apparent that $-8\pi D_0$ is almost exactly equal to $2C_J$. Therefore in the limit as $A \to \infty$ and $C_J \to 0$ the effects of $-8\pi D_0$ in the denominator and $2C_J$ in the numerator will cancel. In this limit the present theory will give a result higher because of $-2C_J$ in the denominator than Eq. (28). This is a smaller discrepancy than that between the lift-coefficient ratios of the present theory and that of Ref. 4.

The three ratios were calculated for $A = 6$ and $A = 3$ over a range of C_J and the results are given in Table 1. Values for $\partial C_L^{(2)}/\partial\alpha$ were taken from Table 1 of Ref. 2 and values for D_0 are from Table 1 of Ref. 5. The ratios of lift coefficients are quite close but, except at the highest values of C_J, rank in the order expected: present theory, Ref. 5, and Ref. 4 at both aspect ratios.

If particular values of α and τ had been chosen and σ had been evaluated and its effect included in the calculations for Table 1, it would have raised the results for the theory of Ref. 4 slightly since it would have been a small positive quantity in each calculation of the ratio. It was neglected because its contribution to the lift coefficient ratio is of higher order in $1/A$ than is considered in the present theory. In this sense, the theory of Ref. 4 is of higher order in $1/A$ than the present one. Extension of the present theory to order $1/A^2$ to permit a higher-order comparison would require, when computing the three-term inner expansion, the solution of a two-dimensional jet-flap problem with induced camber present as complicated as that found by Van Dyke,[7] and such a solution is not available.

Comparison with the results of appropriate experiments would assist in drawing conclusions about the relative accuracy of the three theories. In Figs. 7 and 8 of Ref. 4, its theory (with nonzero σ) for a wing with planform given by Eq. (2) is compared with results from experiments conducted with a rectangular wing at aspect ratios of 6.8 and 2.75 and the agreement is quite good. Comparison with these experimental results has not been made here because the results of the three theories are close enough that the discrepancies between them are less than inaccuracies that could be attributed to using experimental results from a wing with the wrong planform. Nevertheless the closeness of the results given in Table 1 suggests that the agreement between the theory of Ref. 4 and the experimental results confirms the accuracy of all three theories for aspect ratios as low as three. The present theory has the advantage of using the method of matched asymptotic expansions to obtain a result valid for $A \rightarrow \infty$ and throughout the jet-coefficient range of interest. This means that its systematic extension to higher order in $1/A$ will be possible when certain advances in two-dimensional jet-flap theory are made.

Table 1 Ratios of three- to two-dimensional lift coefficients from the present theory and the theories of Refs. 5 and 4

C_J	$\frac{\partial C_L^{(2)}}{\partial\alpha}$ Ref. 2	D_0 Ref. 5	$C_L/C_L^{(2)}$, $A = 6$ Present theory	Ref. 5	Ref. 4, $\sigma = 0$	$C_L/C_L^{(2)}$, $A = 3$ Present theory	Ref. 5	Ref. 4, $\sigma = 0$
0.0	6.283	0.0	0.750	0.750	0.750	0.600	0.600	0.600
0.05	6.468	−0.0040	0.747	0.746	0.743	0.597	0.596	0.592
0.1	6.619	−0.0080	0.746	0.742	0.738	0.595	0.592	0.588
0.2	6.889	−0.0158	0.744	0.737	0.731	0.592	0.588	0.581
0.4	7.370	−0.0318	0.742	0.727	0.720	0.589	0.581	0.572
0.5	7.594	−0.0398	0.741	0.723	0.715	0.588	0.579	0.569
1.0	8.631	−0.0798	0.740	0.707	0.699	0.587	0.570	0.560
1.5	9.595	−0.1198	0.741	0.695	0.688	0.588	0.564	0.556
2.0	10.522	−0.1600	0.743	0.684	0.680	0.591	0.560	0.555
3.0	12.324	−0.2402	0.749	0.668	0.668	0.598	0.555	0.556
4.0	14.097	−0.3204	0.756	0.655	0.659	0.607	0.552	0.556

IV. Conclusions

The method of matched asymptotic expansions has been used to derive a theory for the lift on a high-aspect-ratio wing with jet flap. The derivation has been given for a flat elliptical wing with spanwise-uniform jet coefficient and jet angle. The result is Eq. (26), a simple expression for the lift coefficient.

Since the theory uses velocity components instead of the velocity potential as the dependent variables and thin-airfoil approximations are made throughout it can be regarded as a parallel to the Van Dyke theory derived by using thin-airfoil theory and with the jet flap present. The expansion has not been carried as far; Van Dyke found C_L to order $1/A^2$ by calculating the three-term inner expansion. As occurred in the Van Dyke theory it is not necessary to solve the integral equation for the upwash induced by the lifting surface because the change in lift because of the induced incidence is much smaller than the lift. This simplification makes it possible to avoid the approximations needed in Refs. 4 and 5 in order to calculate C_L. The theory makes use of an equation, Eq. (23), for calculating C_L in terms of w_∞ which was given in these earlier three-dimensional jet-flap theories. The method used here to find w_∞ requires that the expansion be carried to the three-term outer, two-term inner stage in order to calculate C_L to order $1/A$. If the same method were used to calculate C_L to the order, $1/A^2$, to which Van Dyke found it, it would have been necessary to carry the expansion to the four-term outer, three-term inner stage.

The present theory is easily generalized to treat configurations where $c(y)$ is not given by Eq. (2) and α, τ, and C_J are functions of y, as long as the aspect ratio is high and there is no appreciable sweepback. This more general case was treated in Ref. 6 and the result was an expression for C_L which reduces to Eq. (26) for the configuration treated here. Although α_i became a function of y it was possible to use the method of Sec. II to find the two-term inner expansion by adding α_i/A to $W_1 + W_2/A$ since only derivatives with respect to X and Z appear in the two-dimensional flow equations and the jet boundary condition.

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